THE TOPOLATTICE AND PERMUTATION GROUP OF AN INFINITE SET

By

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CHAPTER I

INTRODUCTION

Collections of topologies relative to partial orderings have been studied by many mathematicians (1,2, 3, 4, 5, 6, 7, 8, 9).[1] In particular lattices of topologies on a given set have aroused interest recently (5,8). R. Vaidyanathaswami (8) discussed some of the properties of the complete lattice of all topologies on a set. The purpose of this dissertation is to investigate more thoroughly the properties of the complete lattice, Λ, of all T_1 topologies (topolattice) on an infinite set and to relate the group of permutations of an infinite set S to its topolattice. A certain class of mappings of $\Lambda \otimes \tilde{\Lambda}$ into the two element chain $\{0,1\}$ is found to be isomorphic to the permutation group. The algebraic structure of this group of mappings is investigated.

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[1]Numerals in parentheses refer to the bibliography concluding the dissertation.

CHAPTER II

PRELIMINARY NOTIONS

1. Lattice Theory.

Let P be a set of elements and R a subset of $P \otimes P$. (P,R) is a partially ordered system if the following conditions are satisfied.

(1) The element (a,a) is in R for each a in P.

(2) If (a,b) and (b,a) are in R then a = b.

(3) If (a,b) and (b,c) are in R then (a,c) is in R.

If the relation $\leq$ is defined as follows; $a \leq b$ if and only if (a,b) is in R, then (P,R) may be written $(P,\leq)$.

We say that z is the greatest lower bound (g.l.b.) of $\{x,y\}$, where x, y, and z are in P, if the following condition are satisfied.

(1) $z \leq x$ and $z \leq y$.

(2) For any w in P we have $w \leq z$ whenever $w \leq x$ and $w \leq y$.

The least upper bound (l.u.b.) is defined similarly.

If each pair x,y in P has a l.u.b. and a g.l.b. then the partially ordered system $(P,\leq)$ is said to be a lattice.

In a lattice $(P,\leq)$ two binary operations $\wedge$ and

$\vee$ can be defined as follows; $x \vee y$ is the l.u.b. of $\{x,y\}$ and $x \wedge y$ is the g.l.b. of $\{x,y\}$. These operations have the following properties.

1) $x \vee y = y \vee x$ and $x \wedge y = y \wedge x$ for all x,y (<u>commutativity</u>)
2) $x \wedge x = x$ and $x \vee x = x$ for all x (<u>idempotency</u>)
3) $x \wedge (y \wedge z) = (x \wedge y) \wedge z$ and $x \vee (y \vee z) = (x \vee y) \vee z$ for all x,y,z (<u>associativity</u>)
4) $x \vee y = y$ if and only if $x \wedge y = x$ for all x,y (<u>alternation</u>)

If L is a set of elements in which two binary operations $\vee$ and $\wedge$ are defined for each pair of elements x,y in L having the properties (1) through (4) above, then $(L,\leq)$ is a lattice if $\leq$ is defined by; $x \leq y$ if and only if $x \wedge y = x$. Therefore either set of conditions may be taken as the definition of a lattice.

If in a lattice there is an element o such that $o \leq x$ for all x, then o is called a first element. If there is an element 1 such that $x \leq 1$ for each x, then 1 is called a last element.

A lattice is said to be complete if every set of elements $\{x_\alpha\}$ has a l.u.b., $\bigvee_\alpha x_\alpha$, and a g.l.b., $\bigwedge_\alpha x_\alpha$. A lattice is modular if $x \leq y$ implies that $(x \vee z) \wedge y =$

$x \vee (z \wedge y)$ for all z, distributive if $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$ for all x,y,z, and complemented if there is a first element o and a last element 1 and if for each x there is an element z such that $x \wedge z = o$ and $x \vee z = 1$.

Let $(L, \leq)$ be a lattice. We write

$x \geq y$ whenever $y \leq x$

$x > y$ whenever $x \geq y$ and $x \neq y$

$x < y$ whenever $x \leq y$ and $x \neq y$

$x /\!/ y$ (x and y are incomparable) whenever both $x \leq y$ and $x \geq y$ fail to hold.

We say that y immediately follows x if $x < y$ and if $x \leq z < y$ then $x = z$. If x immediately follows y, then we say that y immediately precedes x.

In a lattice which has a first element, o, any element which immediately follows o is called an atom. If a lattice has a last element, 1, any element which immediately precedes 1 is called a hyperplane.

2. Topology.

A set of elements S with a collection of subsets of S, $\mathcal{U}$, is said to be a T_1 topology on S if the following are true.

1) The null set, $\square$, is in $\mathcal{U}$.
2) If $\{A_\alpha\}$ is any collection of subsets of S with each A_α in $\mathcal{U}$, then $\bigcup_\alpha A_\alpha$ is in $\mathcal{U}$.
3) If $\{A_i\}$ $(i = 1,\ldots n)$ is any finite collection of subsets of S with each set A_i in $\mathcal{U}$, then $\bigcap_{i=1}^{n} A_i$ is in $\mathcal{U}$.
4) If N is a subset of S such that S-N is a finite set, then N is in $\mathcal{U}$.

A set which is null or which has a finite complement will be called an N-set. All other sets are called U-sets. The members of the class $\mathcal{U}$ will be called the open sets under the topology. A set F will be called closed if S-F is open.

A collection of subsets $\mathcal{U}_0$ is a sub-base for the topology $(S,\mathcal{U})$ if each set in $\mathcal{U}$ is the point set sum of sets which are each a finite point set product of sets which are either N-sets or are in $\mathcal{U}_0$. The topologies $(S,\mathcal{U})$ will be denoted by small Greek letters.

If α and β are topologies on S and f is a mapping

of S into itself, then f is said to be continuous in (α, β) if for any set B open under β, $f^{-1}(B)$ is open under α. The mapping, f, is a homeomorphism in (α, β) if it maps S biuniformly onto S and if f is continuous in (α, β) and (β, α).

As examples of topologies we consider the following:

1) Let S be the real line and take the sets $\{x \in S \mid a < x < b\}$ as a sub-basis for the open sets.

2) Let S be any set and let every subset of S be open.

3) Let S be any set and let the null set and only those sets which have finite complements be open.

3. The Topolattice.

A partial ordering among the topologies on a set S may be defined as follows: $\alpha \leq \beta$ if B open under β implies that B is open under α. With this ordering any set of topologies has a g.l.b. and a l.u.b. in the lattice of all T_1 topologies, Λ, on a set S. The g.l.b. of a set of topologies $\{\tau\}$ is the topology obtained by taking all sets $\{B\}$ such that B is open under some τ in $\{\tau\}$ as a sub-base. The l.u.b. is obtained by taking the topology which has for open sets only those which are open under every τ in $\{\tau\}$. The topology φ under which every set is open satisfies $\varphi \leq \tau$ for all τ. The topology λ under which only N-sets are open satisfies $\lambda \geq \tau$ for all τ in Λ. Thus Λ is a complete lattice which has a first and last element.

CHAPTER III

1. Further Theorems on the Topolattice.

If G is any U-set, we define γ_G to be that topology obtained by taking as sub-basis of open sets the class consisting of one member, G. Whenever a symbol γ_G is used it is to be understood that G is a U-set. If, in particular, (s) is the set whose only member is the point s, we write also $\gamma_{(s)} = \tau_s'$. Vaidyanathaswami (8) has shown that each τ_s' is a hyperplane of Λ, but Λ possesses no atoms.

Lemma 1. If H is a U-set open under γ_G, then $H \subset G$.

Proof. Suppose H is a U-set. Now H is the join of finite meets of members of the class consisting of all the N-sets and G. It can be assumed that each member of the join is non-null. If any meet is not contained in G, then H contains a non-null N-set; hence H would be an N-set. Therefore every meet is contained in G. Now H is the join of sets each contained in G, and thus H is contained in G. This proves the lemma.

It may be noted that each member of the join representing H contains all but a finite number of points of G. Consequently G-H is finite.

The symbols in the following definitions will be used throughout to denote the corresponding sets.

$$\Pi(\alpha) = \{\xi \in \Lambda \mid \xi \leq \alpha\}$$
$$\tilde{\Pi}(\alpha) = \{\eta \in \Lambda \mid \eta \geq \alpha\}$$
$$\Phi(\alpha, F) = \{\beta \in \Lambda \mid F(\alpha,\beta) = o\}$$
$$\Psi(\beta, F) = \{\alpha \in \Lambda \mid F(\alpha,\beta) = o\}$$

Where F is a mapping of $\Lambda \otimes \tilde{\Lambda}$ onto $\{o,1\}$.

From this point on, F will be called a chain mapping. A chain mapping F is isotone if $F(\alpha,\beta) \leq F(\alpha_1,\beta_1)$ whenever $(\alpha,\beta) \leq (\alpha_1,\beta_1)$. The latter inequality means $\alpha \leq \alpha_1$ and $\beta \geq \beta_1$.

For $s \in S$ we denote by τ_s the topology under which every set is open except U-sets containing s.

<u>Theorem 1</u>. <u>For each s in S, τ_s and τ'_s are lattice complements, that is, $\tau_s \vee \tau'_s = \lambda$ and $\tau_s \wedge \tau'_s = \varphi$.</u>

<u>Proof</u>. Every set consisting of one point (hence every set) is open under $\tau_s \wedge \tau'_s$. But only N-sets are open under $\tau_s \vee \tau'_s$ since (s) is the only U-set open under τ'_s by Lemma 1 and (s) is not open under τ_s by definition.

The fact that certain hyperplanes exist in Λ but have complements which are not atoms indicates that Λ is non-modular, hence non-distributive. In this connection we have the following.

<u>Remark</u>. <u>If ξ is in Λ and s is in S, then $\xi = (\tau_s \wedge \xi) \vee (\tau'_s \wedge \xi)$. In general, however, equality fails to hold in</u>

$\xi \leq (\tau_s \vee \xi) \wedge (\tau'_s \vee \xi)$.

Proof. On general lattice-theoretic grounds (3), $(\tau_s \vee \xi) \wedge (\tau'_s \vee \xi) \geq \xi \geq (\tau_s \wedge \xi) \vee (\tau'_s \wedge \xi)$. Now suppose G is open under $\tau_s \wedge \xi$ and $\tau'_s \wedge \xi$, but not open under ξ. Then G is a U-set. G-(s) is open under $\tau'_s \wedge \xi$ and, hence is a join of finite meets of sets open under ξ, since such joinands are necessarily U-sets not containing s. Thus G-(s) is open under ξ. Now this means that s is in G since G is not open under ξ. Since G is a join of finite meets of sets open under either τ_s or ξ, some one of these joinands contains s. This joinand is necessarily a U-set (since G is a U-set) and, hence, is not open under τ_s and must be open under ξ. But G is the join of G-(s) and the joinand just discussed so that G is open under ξ which is a contradiction. Thus the first proposition of the remark follows. To show the second, it suffices to consider the example where S is the point set of the Euclidean line, s is any point of S, and ξ is the Euclidean topology of S.

As representation methods for general elements of Λ we prove three theorems, the last of which will be used later in the proof of the main theorem.

Theorem 2. If ξ is in Λ, $\xi = \bigvee_{s \in S} (\xi \wedge \tau_s)$.

Proof. $\xi \geq \bigvee_{s \in S} (\xi \wedge \tau_s)$ since $\xi \geq \xi \wedge \tau_s$ for all s in S.

Suppose that for each s in S, G is open under $\xi \wedge \tau_s$. Then for s in G, there is some subset of G which contains s and is open under ξ, since for each s, G is the join of finite meets of sets open under ξ or τ_s and no U-set which contains s is open under τ_s. The join of such subsets for all s in G is G and is open under ξ. This shows that $\bigvee_s(\xi \wedge \tau_s) \geq \xi$ and the theorem follows.

<u>Theorem 3</u>. <u>If ξ is in Λ and s and t are in S with $s \neq t$, then</u> $\xi = (\xi \vee \tau_s) \wedge (\xi \vee \tau_t)$.

<u>Proof</u>. Clearly, $\xi \leq (\xi \vee \tau_s) \wedge (\xi \vee \tau_t)$. Let G be open under ξ. Then G-(s) is open under $\xi \vee \tau_s$ and G-(t) is open under $\xi \vee \tau_t$. Thus $G = (G-(t)) \cup (G-(s))$ is open under $(\xi \vee \tau_s) \wedge (\xi \vee \tau_t)$, and the theorem is proved.

<u>Theorem 4</u>. (<u>U-representation Theorem</u>). <u>For ξ in Λ, let</u> $\Delta_\xi = \{\gamma_G \in \Lambda \mid \gamma_G \geq \xi\}$. <u>Then</u> $\xi = \bigwedge_{\alpha \in \Delta_\xi} \alpha$.

<u>Proof</u>. Obviously $\xi \leq \bigwedge_{\alpha \in \Delta_\xi} \alpha$. Suppose G is open under ξ. If G is an N-set, then obviously G is open under $\bigwedge_{\alpha \in \Delta_\xi} \alpha$. If G is a U-set, one has $\gamma_G \geq \xi$, since, by definition, γ_G is the maximal topology under which G is open. Thus G is open under $\bigwedge_{\alpha \in \Delta_\xi} \alpha$ since γ_G is in Δ_ξ. This proves the theorem. For ξ in Λ we call $\bigwedge_{\alpha \in \Delta_\xi} \alpha$ the U-representation of ξ. We note that if $\xi \neq \lambda$ then $\Delta_\xi \neq \emptyset$.

2. The Necessary Part of the Main Result

Let f be a permutation (biuniform, onto mapping) of S onto itself. For α, β in Λ we define a mapping F of $\Lambda \otimes \tilde{\Lambda}$ into the two element chain {o,1} as follows:

$F(\alpha,\beta)$ = o if and only if f is continuous in (α,β).

Thus with each permutation f of S we induce a chain mapping (any mapping of $\Lambda \otimes \tilde{\Lambda}$ onto {o,1}) since f is continuous in (φ,φ) but not in (λ,φ). If f and g are permutations of S, then F and G will be used to denote the chain mappings induced by f and g respectively. Whenever the term "induced" is used it will refer to the above definition.

For a chain mapping F we consider the following conditions:

1) F is isotone.
2) For α in Λ there is β in Λ such that $\Psi(\beta;F) = \pi(\alpha)$.
3) For β in Λ there is α in Λ such that $\Phi(\alpha;F) = \tilde{\pi}(\beta)$.

Any mapping satisfying these three properties is called IPIC (isotone, principally ideal chain mapping).

To show that these three conditions are independent (no two imply the third) we cite the following examples:

Let f map S biuniformly onto $S-(s_o)$. Define $F(\alpha,\beta) = o$ if and only if f is continuous in (α,β) and $F(\alpha,\beta) = 1$ otherwise. Clearly F is an isotone chain mapping, for if $(\alpha,\beta) \leq (\alpha_1,\beta_1)$ and $F(\alpha_1,\beta_1) = o$ taking B open under β we have B open under β_1 since $\beta_1 \leq \beta$. But $f^{-1}(B)$ is open under α_1, since f is continuous in (α_1,β_1). Hence $f^{-1}(B)$ is open under α, since $\alpha \leq \alpha_1$. This shows that f is continuous in (α,β), and thus $F(\alpha,\beta) = o$. F satisfies condition (2) but not (3). To see that (2) is satisfied let α be any element in Λ. Let β be the smallest topology under which f(A) is open for each set A which is open under α. To see that (3) is not satisfied we take any topology under which (s_o) is not open and then if α exists as in (3), we would have $F(\alpha,\beta_o) = o$ where $\beta_o < \beta$, hence $\beta_o \not\leq \beta$, by taking $\beta_o = \tau'_{s_o} \wedge \beta$.

To obtain F satisfying (1) and (3) but not (2) we let f be a biuniform mapping of $S-(s_o)$ onto S and proceed in a manner similar to that above. In the following example F satisfies (2) and (3) but not (1).

Take α_o different from φ and λ. For $\beta \| \alpha_o$ we define F' and F" as follows:

1) $\{\gamma \mid F'(\gamma,\beta) = o\} = \{\gamma \mid \gamma \leq \alpha_o \wedge \beta\}$

1') $\{\gamma \mid F''(\beta,\gamma) = o\} = \{\gamma \mid \gamma \geq \alpha_o \vee \beta\}$

If β is ordered with α_o:

2) $\{\gamma \mid F'(\gamma,\beta) = o\} = \{\gamma \mid \gamma \leq \beta\}$

2') $\{\gamma \mid F''(\beta,\gamma) = o\} = \{\gamma \mid \gamma \geq \beta\}$

We show that for any (α,δ) F' and F" have the same value.

<u>Case 1</u>. F' is defined by (1) and F" is defined by (1'). Then $\alpha // \alpha_0$ and $\delta // \alpha_0$.

$F'(\alpha,\delta) = o$ is equivalent to $\alpha \leq \alpha_0 \wedge \delta$, and

$F''(\alpha,\delta) = o$ is equivalent to $\delta \geq \alpha_0 \vee \alpha$.

Thus, $F'(\alpha,\delta) = o$ implies that $\alpha \leq \alpha_0$ so that $F'(\alpha,\delta) = 1$. Also, $F''(\alpha,\delta) = o$ implies that $\delta \geq \alpha_0$ and $F''(\alpha,\delta) = 1$.

<u>Case 2</u>. F' defined by (1) and F" defined by (2'). Then $\delta // \alpha_0$ and

$F'(\alpha,\delta) = o$ is equivalent to $\alpha \leq \alpha_0 \wedge \delta$.

$F''(\alpha,\delta) = o$ is equivalent to $\delta \geq \alpha$.

Now either $\alpha \leq \alpha_0$ or $\alpha \geq \alpha_0$, since F" is defined by (2'). If $\alpha \leq \alpha_0 \wedge \delta$, then $\alpha \leq \delta$. If $\alpha \leq \delta$ and $\alpha \leq \alpha_0$, then $\alpha \leq \alpha_0 \wedge \delta$. One cannot have $\alpha \leq \delta$ and $\alpha \geq \alpha_0$ in this case since this would imply $\delta \geq \alpha_0$. In this case then, $\alpha \leq \alpha_0 \wedge \delta$ is equivalent to $\delta \geq \alpha$. Thus, $F'(\alpha,\delta) = o$ is equivalent to $F''(\alpha,\delta) = o$.

<u>Case 3</u>. F' defined by (2) and F" defined by (1'). Then $\alpha // \alpha_0$ and

$F'(\alpha,\delta) = o$ is equivalent to $\alpha \leq \delta$.

$F''(\alpha,\delta) = o$ is equivalent to $\delta \geq \alpha_0 \vee \alpha$.

We also have $\delta \leq \alpha_0$ or $\delta \geq \alpha_0$. The case $\delta \leq \alpha_0$ (while $\alpha \leq \delta$) is ruled out as before. Thus $\delta \geq \alpha_0$. Now if $\alpha \leq \delta$, then $\alpha \vee \alpha_0 \leq \delta$. If $\delta \geq \alpha_0 \vee \alpha$, then $\alpha \leq \delta$. Thus $\alpha \leq \delta$ is equivalent to $\delta \geq \alpha_0 \vee \alpha$. This completes the proof that F' and F'' agree on $\Lambda \otimes \tilde{\Lambda}$. We define $F(\alpha,\delta) = F'(\alpha,\delta) = F''(\alpha,\delta)$. Clearly F satisfies (2) and (3) by definition. But if $\beta // \alpha_0$, we have by (1');

$$\{\gamma \mid F(\beta,\gamma) = 0\} = \{\gamma \mid \gamma \geq \alpha_0 \vee \beta\},$$

and by (2), since $\alpha_0 \vee \beta > \alpha_0$;

$$\{\gamma \mid F(\gamma,\alpha_0 \vee \beta) = 0\} = \{\gamma \mid \gamma \leq \alpha_0 \vee \beta\}.$$

We shall see later (Lemma 8) that this is impossible for an IPIC mapping. Hence F does not satisfy (1).

The following Lemmas 2 and 3 are implicit in the book of Vaidyanathaswami (8).

<u>Lemma 2</u>. <u>Any induced chain mapping is isotone</u>.

<u>Proof</u>. Suppose $(\alpha_1,\beta_1) \leq (\alpha,\beta)$ and $F(\alpha,\beta) = 0$ where F is induced by a permutation f. Let A be any set open under β_1. Since $\beta_1 \geq \beta$, we have A open under β, and therefore $f^{-1}(A)$ is open under α. Thus $f^{-1}(A)$ is open under α_1 since $\alpha_1 \leq \alpha$. Consequently $F(\alpha_1,\beta_1) = 0$. This proves the lemma.

<u>Lemma 3</u>. <u>If F is an induced chain mapping, then F satisfies conditions</u> (2) <u>and</u> (3).

Proof. Given α, take β to be the topology which has as open sets all sets $f(A)$ where A is open under α and f is the permutation which induces F. If $\gamma \leq \alpha$, then obviously $F(\gamma,\beta) = o$. If $\gamma \not\leq \alpha$, then there is a set A open under α which is not open under γ. Now $f(A)$ is open under β. Thus $F(\gamma,\beta) \neq o$. This proves that F satisfies condition (2). Similarly one may show that F satisfies (3).

Combining Lemmas 2 and 3 we have

Theorem 5. Any induced chain mapping is IPIC.

Lemma 4. If F is induced by a permutation f and if G is a U-set with $f(G) = H$, then

$$\Psi(\tau_{f(s)};F) = \Pi(\tau_s),\ \Psi(\tau'_{f(s)};F) = \Pi(\tau'_s)$$
$$\Phi(\tau_s;F) = \widetilde{\Pi}(\tau_{f(s)}),\ \Phi(\tau'_s;F) = \widetilde{\Pi}(\tau'_{f(s)});$$

for all $s \in S$, and $\Phi(\gamma_G;F) = \widetilde{\Pi}(\gamma_H)$, $\Psi(\gamma_H;F) = \Pi(\gamma_G)$.

Proof. The lemma follows immediately from the fact that $\Psi(\beta;F) = \Pi(\alpha)$ and $\Phi(\alpha;F) = \widetilde{\Pi}(\beta)$ where A is open under α if and only if $f(A)$ is open under β. This was noted in the proof of Lemma 3.

3. Sufficient Part of the Main Result

Throughout this section, K will denote an arbitrary but fixed IPIC mapping.

Lemma 5. *If* $\gamma \leq \alpha$, $\Phi(\alpha;K) \subset \Phi(\gamma;K)$. *If* $\delta \leq \beta$, $\Psi(\delta;K) \subset \Psi(\beta;K)$.

Proof. Suppose $\gamma \leq \alpha$ and θ is in $\Phi(\alpha;K)$. Then $K(\alpha,\theta) = o$. But since K is isotone and $(\gamma,\theta) \leq (\alpha,\theta)$, we have $K(\gamma,\theta) = o$ so that θ is in $\Phi(\gamma;K)$. For the second part suppose $\delta \leq \beta$ and select θ in $\Psi(\delta;K)$. Then $K(\theta,\delta) = o$ and since $(\theta,\beta) \leq (\theta,\delta)$, we have $K(\theta,\beta) = o$, which implies that θ is in $\Psi(\beta;K)$.

Lemma 6. *For* α *in* Λ *there is an element* β *in* Λ *such that* $\Phi(\alpha;K) = \tilde{\Pi}(\beta)$.

Proof. From (2) there is an element β in Λ such that $\Psi(\beta;K) = \Pi(\alpha)$. From (3) there is γ in Λ such that $\Phi(\gamma;K) = \tilde{\Pi}(\beta)$. Thus $K(\gamma,\beta) = o$. Then γ is in $\Psi(\beta;K) = \Pi(\alpha)$. Hence $\gamma \leq \alpha$. Let θ be any element in $\Phi(\gamma;K)$. Then $\theta \geq \beta$. But $K(\alpha,\beta) = o$, so that $K(\alpha,\theta) = o$ and θ is in $\Phi(\alpha;K)$. Therefore $\Phi(\gamma;K) \subset \Phi(\alpha;K)$. This with Lemma 5 shows that $\Phi(\gamma;K) = \Phi(\alpha;K) = \tilde{\Pi}(\beta)$.

By a dual proof one obtains

Lemma 7. *For* β *in* Λ *there is an element* α *in* Λ *such that* $\Psi(\beta;K) = \Pi(\alpha)$.

<u>Lemma 8</u>. (<u>Fundamental Lemma</u>) $\Phi(\alpha;K) = \tilde{\Pi}(\beta)$ <u>is equivalent to</u> $\Psi(\beta;K) = \Pi(\alpha)$.

<u>Proof</u>. Suppose $\Phi(\alpha;K) = \Pi(\beta)$. From (2) there is δ in Λ with $\Psi(\delta;K) = \Pi(\alpha)$. Thus $K(\alpha,\delta) = o$. But $\Phi(\alpha;K) = \tilde{\Pi}(\beta)$. Therefore $\delta \geq \beta$ and $\Psi(\beta;K) \subset \Pi(\alpha)$, by Lemma 5. By Lemma 7 we know that $\Psi(\beta;K)$ is a principal ideal in Λ and hence $\alpha \in \Psi(\beta;K)$, since $\Phi(\alpha;K) = \tilde{\Pi}(\beta)$. Thus $\Psi(\beta;K) = \Pi(\alpha)$. The reverse implication of Lemma 8 follows dually.

<u>Lemma 9</u>. <u>If</u> $\Phi(\alpha;K) = \Phi(\gamma;K)$, <u>then</u> $\alpha = \gamma$. <u>If</u> $\Psi(\beta;K) = \Psi(\delta;K)$, <u>then</u> $\beta = \delta$.

<u>Proof</u>. Assume that $\Phi(\alpha;K) = \Phi(\gamma;K)$. By Lemma 6 there are elements β and δ such that $\Phi(\alpha;K) = \tilde{\Pi}(\beta)$ and $\Phi(\gamma;K) = \tilde{\Pi}(\delta)$. But $\tilde{\Pi}(\beta) = \tilde{\Pi}(\delta)$ implies that $\beta = \delta$ on purely lattice-theoretic grounds. From Lemma 8 we have $\Psi(\beta;K) = \Pi(\alpha) = \Pi(\gamma)$ which implies that $\alpha = \gamma$. We obtain the second result of Lemma 9 by a dual proof.

<u>Lemma 10</u>. <u>For each</u> s <u>in</u> S, <u>there is a point</u> t <u>in</u> S such <u>that</u> $\Phi(\tau'_s;K) = \tilde{\Pi}(\tau'_t)$. <u>For each</u> t <u>in</u> S <u>there is a point</u> s <u>in</u> S <u>such that</u> $\Psi(\tau'_t;K) = \Pi(\tau'_s)$.

<u>Proof</u>. From Lemma 6 there is an element β in Λ such that $\Phi(\tau'_s;K) = \tilde{\Pi}(\beta)$ and, by Lemma 8, $\Psi(\beta;K) = \Pi(\tau'_s)$. We divide the proof into the following three

mutually exclusive cases:

<u>Case 1</u>. $\beta = \lambda$. Now λ is in $\Phi(\alpha;K)$ for all α in Λ since $\Phi(\alpha;K)$ is a principal dual ideal in Λ. Thus, from Lemma 8, we have $\Psi(\lambda;K) = \Lambda$. But in this case $\Psi(\lambda;K) = \pi(\tau_s') \neq \Lambda$ by Lemma 8. Consequently Case 1 is impossible.

<u>Case 2</u>. For each $t \in S$, $\beta \parallel \tau_t'$. In this case no non-null finite sets are open under β since no set consisting of one point is open under β. Since $\beta \neq \lambda$, there is an infinite U-set open under β. Let G be the infinite U-set obtained by deleting a point from this set open under β. By Lemma 1, $\beta < \gamma_G$. Thus $\Psi(\beta;K) \subsetneq \Psi(\gamma_G;K)$ by Lemmas 5 and 9. But $\Psi(\beta;K) = \pi(\tau_s')$, by Lemma 8, so that $\pi(\tau_s') \subsetneq \Psi(\gamma_G;K)$; however, since $\Psi(\gamma_G;K)$ is a principal ideal in Λ, we have $\Psi(\gamma_G;K) = \Lambda$, because τ_s' is a hyperplane. This in turn implies, by Lemma 9, that $\gamma_G = \lambda$ which is false. Thus we see that Case 2 is also impossible. The only remaining possibility is

<u>Case 3</u>. For some $t \in S$, $\beta \leq \tau_t'$. Suppose that $\beta < \tau_t'$. Then $\Psi(\beta;K) \subsetneq \Psi(\tau_t';K)$, by Lemmas 5 and 9. Thus, since $\pi(\tau_s') = \Psi(\beta;K)$, we have $\pi(\tau_s') \subsetneq \Psi(\tau_t';K)$. But this implies that $\Psi(\tau_t';K) = \Lambda$ and, hence, that $\tau_t' = \lambda$, which is false. Thus $\beta = \tau_t'$ for some t in S and $\Phi(\tau_s';K) = \tilde{\pi}(\tau_t')$. The second statement of Lemma 10 follows dually.

<u>Lemma 11</u>. <u>The mapping</u> $f : S \to S$ <u>established by</u> $f(s) = t$, <u>where</u> $\Phi(\tau'_s;K) = \tilde{\Pi}(\tau'_t)$, <u>is a permutation of</u> S.

<u>Proof</u>. This lemma follows immediately from Lemmas 8, 9, and 10.

<u>Lemma 12</u>. $\Phi(\tau_s;K) = \tilde{\Pi}(\tau_{f(s)})$ <u>and</u> $\Psi(\tau_{f(s)};K) = \Pi(\tau_s)$.

<u>Proof</u>. Clearly $\tau_s < \tau_t$ for all t in S with $t \neq s$. Hence $\Phi(\tau_s;K) \supset \Phi(\tau'_t;K)$ for $t \neq s$. Therefore $\Phi(\tau_s;K) \supset \bigcup_{t \neq s} \tilde{\Pi}(\tau'_{f(t)})$ by Lemma 11. But $\Phi(\tau_s;K)$ is a principal dual ideal in Λ and, hence, $\bigwedge_{t \neq s} \tau'_{f(t)} = \bigwedge_{x \neq f(s)} \tau'_x$ is in $\Phi(\tau_s;K)$. Also, $\tau_{f(s)} = \bigwedge_{x \neq f(s)} \tau'_x$, so $\tilde{\Pi}(\tau_{f(s)}) \subset \Phi(\tau_s;K)$. From (3) there is α such that $\Phi(\alpha;K) = \tilde{\Pi}(\tau_{f(s)})$. Thus, $K(\alpha, \tau'_{f(x)}) = o$ for $x \neq s$, since $\tau'_{f(x)} > \tau_{f(s)}$, for $x \neq s$, and K is isotone. This implies that $\alpha \leq \tau'_x$ for $x \neq s$, by Lemma 10. Since, again, $\tau_s = \bigwedge_{x \neq s} \tau'_x$, $\alpha \leq \tau_s$. Then $\Phi(\alpha;K) \supset \Phi(\tau_s;K)$. This concludes the proof of the first statement of Lemma 12. The second statement follows by the Fundamental Lemma.

<u>Lemma 13</u>. <u>Let</u> H <u>be any</u> <u>U-set</u>. <u>Then</u> $\Psi(\gamma_H;K) = \Psi(\gamma_H;F)$ <u>where</u> F <u>is induced by the permutation</u> f <u>of Lemma 11</u>.

<u>Proof</u>. Let $\Psi(\gamma_H;K) = \pi(\alpha)$ in accordance with Lemma 7. Let $G = f^{-1}(H)$. Then $\Psi(\gamma_H;F) = \pi(\gamma_G)$ by Lemma 4. By Lemmas 4 and 12, $\Phi(\tau_x;K) = \Phi(\tau_x;F)$, so that $K(\tau_x,\gamma_H) = F(\tau_x,\gamma_H)$ for all x in S. Thus $\tau_x \leq \alpha$ is equivalent to $\tau_x \leq \gamma_G$. Since x in G is equivalent to $\tau_x \not\leq \gamma_G$ and x not in G is equivalent to $\tau_x \leq \gamma_G$, by Lemma 1, we have

($\top$) x is in G if and only if $\tau_x \not\leq \alpha$, and

($\top\top$) x is not in G if and only if $\tau_x \leq \alpha$.

By ($\top$), each point x of G is contained in a U-set A_x open under α, but, by ($\top\top$), such a set A_x is contained in G. Hence $G = \bigcup_{x \in G} A_x$ is open under α and $\alpha \leq \gamma_G$. We note again by ($\top\top$) that the only U-sets open under α are subsets of G. We also note that if G is finite, then $\alpha \leq \gamma_G$ implies that $\alpha = \gamma_G$. Therefore for the remainder of the proof it suffices to assume that G is infinite. Let β be such that $\Phi(\gamma_G;K) = \tilde{\pi}(\beta)$. Now, since $K(\gamma_G,\beta) = o$ and $\alpha \leq \gamma_G$, we have $K(\alpha,\beta) = o$. Hence $\beta \geq \gamma_H$, since $\Phi(\alpha;K) = \tilde{\pi}(\gamma_H)$. (This follows from Lemma 8 and the fact that $\Psi(\gamma_H;K) = \pi(\alpha)$.) Now $\beta \geq \gamma_H$ implies that $\beta = \gamma_B$ for some U-set B contained in H with H-B finite (since $\beta \neq \lambda$). If B = H, then the lemma follows. Suppose $H-B \neq \square$. We let $B_1 = B \cup (x)$ where

$x \in H-B$. Then γ_{B_1} immediately precedes γ_B. Hence letting $\Psi(\gamma_{B_1};K) = \pi(\alpha_1)$ we have (remembering that $\Psi(\gamma_B;K) = \pi(\gamma_G))$ α_1 immediately precedes γ_G. Now any U-set open under α_1 is contained in G since $\alpha \leq \alpha_1$. Thus $\alpha_1 = \gamma_G \wedge \tau'_y$ for some y in G, this being the only way α_1 can immediately precede γ_G and satisfy the condition that any U-set open under α_1 is contained in G. Thus $\alpha_1 \leq \tau'_y$. But, since $\Psi(\tau'_{f(y)};K) = \pi(\tau'_y)$ and $\Phi(\alpha_1;K) = \tilde{\pi}(\gamma_{B_1})$, we have $K(\alpha_1, \tau'_{f(y)}) = o$ and, hence, $\tau'_{f(y)} \geq \gamma_{B_1}$. This is impossible unless B_1 is finite. If B_1 is finite, then G is finite. This contradicts the assumption that $H-B \neq 0$ and establishes the lemma.

<u>Lemma 14</u>. <u>For any</u> $\tau \in \Lambda$, $\Phi(\tau;K) = \Phi(\tau;F)$ <u>and</u> $\Psi(\tau;K) = \Psi(\tau;F)$. <u>Thus</u>, K <u>and</u> F <u>are identical</u>.

<u>Proof</u>. Let $\Phi(\tau;K) = \tilde{\pi}(\alpha)$ and $\Phi(\tau,F) = \tilde{\pi}(\beta)$ in accordance with Lemma 6. By Lemma 13 $\gamma_G \geq \alpha$ is equivalent to $\gamma_G \geq \beta$. Then, by the U-representation of Theorem 4, $\alpha = \beta$. From the Fundamental Lemma 8, we have $\Psi(\tau;K) = \Psi(\tau;F)$.

CHAPTER IV

THE MAIN THEOREM AND A GROUP STRUCTURE FOR IPIC MAPPINGS

Theorem 6. A chain mapping $K : \Lambda \otimes \tilde{\Lambda} \longrightarrow \{0,1\}$ is induced by a permutation of S if and only if K is IPIC.

Proof. The theorem follows immediately from Theorem 5 and Lemmas 11 and 14.

We may restate Theorem 6 as

Theorem 7. Inducement is a biuniform correspondence between the permutation group of S and the set of all IPIC mappings of $\Lambda \otimes \tilde{\Lambda}$.

We denote by $\mathcal{M}$ the class of IPIC mappings of $\Lambda \otimes \tilde{\Lambda}$ and by $\mathcal{L}$ the permutation group of S.

Since $\mathcal{L}$ is a group in biuniform correspondence with $\mathcal{M}$, we may impose the group structure of $\mathcal{L}$ on $\mathcal{M}$. We find

Lemma 15. Let f,g be in $\mathcal{L}$ and h = fg (g followed by f). Then $H(\alpha,\beta) = 0$ if and only if there is $\gamma \in \Lambda$ with $F(\gamma,\beta) = G(\alpha,\gamma) = 0$.

Proof. The "if" part is well known. Suppose $H(\alpha,\beta) = 0$. Let γ be the topology under which A is open if and only if f(A) is open under β. Then $F(\gamma,\beta) = 0$. Also, if A is open under γ, f(A) is open under β and

$(fg)^{-1}(f(A)) = g^{-1}(f^{-1}(f(A))) = g^{-1}(A)$ is open under α. Hence, $G(\alpha,\delta) = o$.

We may restate Lemma 15, noting that $\{o,1\}$ is a Boolean Algebra, as

<u>Theorem 8</u>. <u>For</u> f,g <u>in</u> $\mathcal{L}$, $h = fg$: $H(\alpha,\beta) = \bigwedge_{\gamma\in\Lambda}(F(\gamma,\beta)\vee G(\alpha,\gamma))$.

Thus, group multiplication in $\mathcal{M}$ may be regarded as a sort of "lattice-theoretic dual convolution."

<u>Lemma 16</u>. <u>For</u> f <u>in</u> $\mathcal{L}$, $f = e$ (<u>identity mapping of</u> S) <u>if and only if</u> $F(\alpha,\alpha) = o$; <u>for all</u> α <u>in</u> Λ.

<u>Proof</u>. The "only if" part is obvious. Suppose $F(\alpha,\alpha) = o$, for all $\alpha\in\Lambda$, and $f(s)\neq s$ for some s in S. Then $F(\tau_s,\tau_s) \neq o$ contrary to assumption, since $(f(s))$ is open in τ_s, but $(f^{-1}(f(s))) = (s)$ is not open in τ_s.

<u>Lemma 17</u>. <u>Let</u> F <u>be in</u> $\mathcal{M}$. <u>Then</u> G <u>in</u> $\mathcal{M}$ <u>is the inverse under dual convolution of</u> f <u>if and only if there is a pair of mappings</u>:

$$\gamma : \Lambda \longrightarrow \Lambda, \quad \delta : \Lambda \longrightarrow \Lambda$$

<u>so that for</u> α <u>in</u> Λ, $F(\alpha,\gamma(\alpha)) = G(\gamma(\alpha),\alpha) = o$ <u>and/or</u> $F(\delta(\alpha),\alpha) = G(\alpha,\delta(\alpha)) = o$.

<u>Proof</u>. G is the inverse of F if and only if their products in both orders equal the member of $\mathcal{M}$ associated with e; however, the equality of this product in either order with the member of $\mathcal{M}$ associated with e is sufficient.

We may summarize the results of this section as

<u>Theorem 9</u>. <u>Inducement is a biuniform mapping between</u> $\mathcal{L}$ <u>and</u> $\mathcal{M}$. $\mathcal{M}$ <u>forms a group under dual convolutions and the identity of this group is the member of</u> $\mathcal{M}$ <u>which vanishes on the diagonal of</u> $\Lambda \otimes \tilde{\Lambda}$.

CHAPTER V

FURTHER QUESTIONS

A mapping of Λ onto itself which preserves order is called an automorphism. It is well known that the set of all automorphisms on a lattice forms a group (a sub-group of the transformation group). It has been shown by Professor David Ellis (10) that the group of automorphisms of Λ is isomorphic to the permutation group of S and hence isomorphic to the group of IPIC mappings on $\Lambda \otimes \tilde{\Lambda}$. This gives some hold on Λ, leading to the question: What are necessary and sufficient conditions on an arbitrary lattice that it be isomorphic to Λ?

Many lattice-theoretic questions concerning Λ arise. As we mentioned before, Λ is not modular, hence not distributive. A characterization of the modular sub-lattices and of the distributive sub-lattices may shed some light on the question stated above. The same applies to complemented elements and complemented sub-lattices. A complemented sub-lattice is a sub-lattice which contains the complement of each of its elements.

Since the U-representation theorem played such an important role in the proof of the main result of the dissertation, it seems likely that other representations

in Λ will find use in the future.

Professor David Ellis has shown that the IPIC group of any lattice with first and last elements is isomorphic to the group of automorphisms of the lattice. This also contributes to the general problem of characterizing Λ among general lattices.

It would be of interest to determine the distribution of topologies in Λ which have additional properties such as; Hausdorff, bicompact, Hausdorff and bicompact, or metrizable.

We define a Galois connection between the collection of sub-lattices of Λ and the collection of subgroups of the permutation group $\mathcal{L}$ of S as follows: If A is a sub-lattice of Λ, then A^* is the subgroup of all permutation of $\mathcal{L}$ which are homeomorphic in (α,α) for each α in A. If B is a subgroup of $\mathcal{L}$, then B^+ is the sub-lattice of all the elements α of Λ such that each permutation of B is homeomorphic in (α,α). A natural question here is what are the closed elements (those sets A such that $A^{*+} = A$ and sets B such that $B^{+*} = B$).

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BIOGRAPHICAL SKETCH

Robert Waller Bagley was born April 14, 1921 in Copiah County, Mississippi. He attended public schools in Jackson, Mississippi and Wyandotte, Michigan. In 1939 he graduated from Roosevelt High School in Wyandotte. He entered the University of Michigan in 1941, withdrawing in 1943 to enlist in the United States Army Air Corps. Upon his discharge in 1945, he re-entered the University of Michigan and received the Bachelor of Science degree from that institution in 1947 and the Master of Science degree in 1949 at Tulane University. While at Tulane, he held a graduate assistantship and was an instructor for the Summer Session 1948. He served as an instructor at Florida State University for the 1949-50 sessions. He studied at the University of Texas during the 1950-51 terms. In 1952 he entered the University of Florida and there completed the work for the Doctor of Philosophy degree in 1954. At the University of Florida he was a graduate assistant and interim instructor.

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